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The cohomology groups of degree 3 of Siegel modular varieties of genus 2

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§0. Introduction.

We discuss two types of modular symbols on Siegel modular varieties of genus 2: One is holomorphic, another totally real. In §1, we give a simple result for the first case. In §2, on the second case we discuss a conjecture on its Hodge type, and explain how to reduce it to another conjectures on L -functions.

Notation. $Sp(2; \mathbb{R})$: the real symplectic group of rank 2.

$Sp(2; \mathbb{Z})$: the integral symplectic subgroup in $Sp(2; \mathbb{R})$. l : natural number ≥ 3 .

$\Gamma(l)$: the principal congruence subgroup of $Sp(2; \mathbb{Z})$. \mathfrak{h}_2 : the Siegel upper half space of genus 2. $V = \Gamma(l) \backslash \mathfrak{h}_2$: the quotient of \mathfrak{h}_2 by $\Gamma(l)$, which is a smooth algebraic variety over \mathbb{C} . \tilde{V} : a smooth toroidal compactification of V .

§1. Modular subvarieties and coniveau filtration.

(1.1) Coniveau filtration.

Let X be a smooth compact algebraic variety over \mathbb{C} with dimension n . Let Y be a subvariety of dimension m in X . Then by restriction, we have a natural map of Hodge structures: $H^k(X, \mathbb{Q}) \rightarrow H^k(Y, \mathbb{Q})$. If Y is smooth, then by Poincaré duality, we have $H^{2m-k}(Y, \mathbb{Q})(m) \rightarrow H^{2n-k}(X, \mathbb{Q})(n)$. Or, equivalently, we have $H^i(Y, \mathbb{Q})(-d) \rightarrow H^{i+2d}(X, \mathbb{Q})$. Here (m) etc. are Tate twists. In general, we have an exact sequence of local cohomology

$$\rightarrow H_Y^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q}) \rightarrow H^i(X-Y, \mathbb{Q}) \rightarrow \dots$$

By Deligne, this is an exact sequence of mixed Hodge structures.

When Y is smooth, we have $H_Y^i(X, \mathbb{Q}) \cong H^{i-2d}(Y, \mathbb{Q})(-d)$ by Gysin isomorphism.

Definition-Proposition. The rational sub-Hodge structure of $H^i(X, \mathbb{Q})$

defined by

$$F^d H^i(X, \mathbb{Q}) = \sum_{\substack{Y \subset X \\ \text{codimension } d}} \text{Im} \{ H_Y^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q}) \}$$

has Hodge type $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a+b=i, a \geq d, b \geq d\}$.

The above filtration is called "coniveau" filtration by Grothendieck. It is assumed that this filtration has a crucial role in the arithmetic of cycles on X .

(1.2) The "boundary part" of Hodge structure $H^3(\tilde{V}, \mathbb{Q})$.

Let \tilde{V} be a smooth toroidal compactification of V along the cusps. Put $D = \tilde{V} - V$ and $D = \bigcup_i D_i$ the decomposition into irreducible components D_i . Since each D_i is smooth in \tilde{V} , the restriction map $H^3(\tilde{V}, \mathbb{Q}) \rightarrow H^3(D_i, \mathbb{Q})$ define a morphism of rational Hodge structures $p_{D_i}: H^1(D_i, \mathbb{Q})(-1) \rightarrow H^3(\tilde{V}, \mathbb{Q})$.

Definition. $H^3(M_\infty, \mathbb{Q}) \stackrel{\text{def}}{=} \sum_i \text{Im } p_{D_i}$.

$H^3(M_\infty, \mathbb{Q})$ is a polarized sub-Hodge structure with Hodge type $\{(2,1), (1,2)\}$.

If we take the Igusa model for \tilde{V} as [Yamazaki], each D_i is an elliptic modular surface over a modular curve C_i . Then $H^1(D_i, \mathbb{Q}) \cong H^1(C_i, \mathbb{Q})$.

(1.3) Hilbert modular surfaces.

By modular embedding, or Satake embedding, a Hilbert modular surface S is mapped to V : $S \xrightarrow{f} V$. We can compactify $f: \tilde{S} \xrightarrow{\tilde{f}} \tilde{V}$. Since $b_1(\tilde{S}) = 0$ for any Hilbert modular surface, $H^1(\tilde{S}, \mathbb{Q})(-1) \rightarrow H^3(\tilde{V}, \mathbb{Q})$ is zero map.

(1.4) "Diagonal" embedding.

Let $(z_1, z_2) \in H \times H \rightarrow \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \in \mathcal{G}_2$ be the holomorphic

map from the product of the upper half plane H to h_2 . This induces a holomorphic map from a product $E_\alpha = R_1 \times R_2$ of elliptic modular curves R_i ($i=1,2$) to $V : f_{E_\alpha} : E_\alpha = R_1 \times R_2 \rightarrow V$.

By K nneth formula,

$$H^1(\tilde{E}_\alpha, \mathbb{Q}) = H^1(\tilde{R}_1, \mathbb{Q}) \oplus H^1(\tilde{R}_2, \mathbb{Q}).$$

In general, $H^1(\tilde{R}_i, \mathbb{Q}) \neq \{0\}$, hence $H^1(\tilde{E}_\alpha, \mathbb{Q}) \neq \{0\}$.

(1.5) Proposition. Consider the Poincar  dual $\rho_{E_\alpha} : H^1(\tilde{E}_\alpha, \mathbb{Q}(-1)) \rightarrow H^3(\tilde{V}, \mathbb{Q})$ of the restriction map $H^3(\tilde{V}, \mathbb{Q}) \rightarrow H^3(\tilde{E}_\alpha, \mathbb{Q})$. Then $\text{Im } \rho_\alpha$ is contained in $H^3(M_\infty, \mathbb{Q})$.

(1.6) Proof of Proposition

Recall that $H^1(\tilde{V}, \mathbb{Q}) = \{0\}$. Hence the intersection form

$$\Phi : H^3(\tilde{V}, \mathbb{Q}) \times H^3(\tilde{V}, \mathbb{Q}) \longrightarrow \mathbb{Q}(-3)$$

is non-degenerate, and its restriction to $H^3(M_\infty, \mathbb{Q})$ is also non-degenerate.

Therefore, if $\text{Im } \rho_{E_\alpha} \not\subset H^3(M_\infty, \mathbb{Q})$ is true, then there exists a cohomology class $\xi \neq 0 \in \text{Im } \rho_{E_\alpha}$ such that for any $\eta \in H^3(M_\infty, \mathbb{Q})$ $\Phi(\xi, \eta) = 0$. Thus it suffices to show the following.

(1.7). Lemma. For any $\xi \neq 0$ in $\text{Im } \rho_{E_\alpha}$, there exists

$\eta \in H^3(M_\infty, \mathbb{Q})$ such that $\Phi(\xi, \eta) \neq 0$.

In order to prove the above Lemma, it is necessary to write the intersection number $\Phi(\xi, \eta)$ in terms of

$$\xi' \in H^1(\tilde{E}_\alpha, \mathbb{Q})(-1) \text{ and } \eta' \in H^1(\mathbb{D}_i, \mathbb{Q})(-1),$$

if we choose ξ', η' by

$$\rho_{E_\alpha}(\xi') = \xi, \quad \rho_{\mathbb{D}_i}(\eta') = \eta.$$

(1.8) Lemma Let Y be the intersection of \tilde{E}_α and \mathbb{D}_i .

Y is possibly empty. Let us consider the morphisms by restriction:

$$\begin{cases} r_1 : H^1(\tilde{E}_\alpha, \mathbb{Q})(-1) \rightarrow H^1(Y, \mathbb{Q})(-1); \\ r_2 : H^1(\mathbb{D}_i, \mathbb{Q})(-1) \rightarrow H^1(Y, \mathbb{Q})(-1). \end{cases}$$

Then

$$\Phi(\xi, \eta) = \Phi_Y(r_1(\xi'), r_2(\eta')).$$

Here Φ_Y is the intersection form of the curve Y :

$$\Phi_Y : H^1(Y, \mathbb{Q}) \times H^1(Y, \mathbb{Q}) \rightarrow \mathbb{Q}(-1).$$

The above Lemma is a standard fact in the intersection theory, and may be found in the text book of [Fulton].

Thus the proof of Lemma (1.7) is reduced to the following

(1.9) Lemma For any $\xi \neq 0$ in $\text{Im } \rho_{E_\alpha}$, there exists an element $\eta' \in H^1(\mathbb{D}_i, \mathbb{Q})$ of some irreducible component \mathbb{D}_i of \mathbb{D} such that for $Y = \mathbb{D}_i \cap \tilde{E}_\alpha$, $\Phi_Y(r_1(\xi'), r_2(\eta')) \neq 0$.

This Lemma is immediate from the following result of [Yamazaki].

(1.10) Proposition. For $\tilde{E}_\alpha = \tilde{R}_1 \times \tilde{R}_2 \rightarrow \tilde{V}$, there exists an irreducible component \mathcal{D}_i of \mathcal{D} such that for $Y = \mathcal{D}_i \cap \tilde{E}_\alpha$,

$$H^1(\mathcal{D}_i, \mathbb{Q}) \cong H^1(Y, \mathbb{Q}).$$

This completes the proof of Proposition (1.5).

§2. Conjectures on totally real modular symbols

(2.1) In the construction of L -functions associated to Siegel modular forms of genus 2, Andrianov considered a kind of totally real embedding of hyperbolic three spaces into the Siegel upper half space of degree 2.

K is an imaginary quadratic field, $SL_2(\mathcal{O}_K)$ is the special linear group with entries in the ring of integers \mathcal{O}_K of K , which is a discrete subgroup of $SL_2(\mathbb{C})$. For some congruence subgroup Γ' of $SL_2(\mathcal{O}_K)$, there exists a map

$$m: \Gamma' \backslash SL_2(\mathbb{C}) / K' \longrightarrow \Gamma(2) \backslash Sp(2; \mathbb{R}) / K$$

induced from an injective homomorphism $SL_2(\mathbb{C}) \rightarrow Sp(2; \mathbb{R})$.

Here K' and K are maximal compact subgroups of $SL_2(\mathbb{C})$ and $Sp(2; \mathbb{R})$, respectively.

Let \bar{V} be the Satake compactification of $V = \Gamma(2) \backslash G_2$.
Then the canonical compactification of m defines an element in $H_3(\bar{V}; \mathbb{Z})$. More careful investigation shows the following.

(2.2) Lemma. \bar{m} defines an element $\sum_{[\bar{m}]} \text{in } IH_3(\bar{V}; \mathbb{Z})$.

Here $IH_3(\bar{V}; \mathbb{Z})$ is the intersection homology group of degree 3 of \bar{V} with middle perversity.

We denote by $[\bar{m}]^* \in IH^3(\bar{V}; \mathbb{Q})$ the Poincaré dual of the image of the fundamental class $[\bar{m}]$ of $\Gamma' \backslash SL_2(\mathbb{O}_K) / K'$.

The cohomology group $IH^3(\bar{V}; \mathbb{Q})$ has a Hodge structure defined by Morihiko Saito.

(2.3) Conjecture. The cycle $[\bar{m}]^* \in IH^3(\bar{V}, \mathbb{Q})$ has Hodge type $\{(3, 0); (0, 3)\}$.

The above conjecture is reduced to the following conjectures for L functions for harmonic forms on V .

(2.4) Conjecture. Let ω be a L^2 -harmonic form on V of type (2.1). Assume that ω is an common eigen-form of all Hecke operators. Then,

(i) Let $L(s, \omega)$ be the L-function for Spinor representation associated to ω . Then $L(s, \omega)$ has a possibly simple pole at $s=2$, &

$$\operatorname{Res}_{s=2} L(s, \omega) = \int_{\gamma} \omega,$$

with γ is a linear combination of cycles $[\bar{m}] \in [H_3(\bar{V}; \mathbb{Z})]$.

(ii) $L(s, \omega)$ is an entire function if it is multiplied with the Γ -factor $\Gamma(s)\Gamma(s-1)$.

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